Government Debt and Default in a Minimal State

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Abstract

I construct a model of a small open economy in which government spending is necessary to mitigate transaction cost. This provides a simple raison d’être for a government and generates features many sovereign default models do not have: taxes and government spending. Even though the government sector is relatively small, the model can generate average and peak levels of government debt as well as second moments in line with business cycle statistics of Argentina. The model is solved using an algorithm that works roughly similar to earlier works in this literature, but avoids their issues with erroneous approximations.

JEL Codes: F32, F34, H63

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1 Motivation and Summary

Recent stochastic general equilibrium models of small open economies that may undergo a sovereign default, namely Aguiar and Gopinath (2006), Arellano (2008), Cuadra and Sapriza (2008), Hatchondo et al. (2009), Yue (2010), Lizarazo (2009, 2013) and Mendoza and Yue (2012) provide a microeconomic framework that helps to understand the quantitative properties of risk premia on sovereign debt and of output, private consumption and the trade balance of such economies. While Aguiar and Gopinath (2006), Arellano (2008), Yue (2010) and Lizarazo (2009, 2013) present “bare bone” models without much detail apart from the interaction of lenders and government(s) in a stochastic environment, other researchers modeled changing governments (as Cuadra and Sapriza, 2008, Hatchondo et al., 2009) and even a full fledged economy with labor providing households and a firm sector (as Mendoza and Yue, 2012). However, even though the focus of interest of these works is an action of the government (its choice to default), the

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latter is nonexistent in all models except for its debt. This makes it hard to see why a government exists at all – a banking sector could also provide consumption smoothing for households.

This paper provides a model where government has an explicit role: It provides the services and institutions necessary to mitigate transaction cost. The more transactions, the larger the need for services that help to seamlessly conduct those transactions. Government debt is used to adjust the tax charged to finance these measures such that private consumption is smoothed. This purely neoclassical role of the government means that its spending is a complement to other factors, implying pro cyclical optimal government spending. While the scope of activity of modern governments in the rich world is obviously much larger, both the role of government in the USA during the 19th century and in some emerging economies, particularly Argentina, can be characterized this way. For both, the correlation coefficient of trend deviations of real GDP and trend deviations of real government spending is indeed positive: for the USA 1870–1900 it is 0.36, for Argentina 1968–2011 it is 0.66. In contrast, the correlation in recent data is -0.39 for the USA, -0.24 for France and -0.34 for Germany. Since Argentina underwent three defaults in the last 100 years and offers a set of relatively long time series, most of the works cited above and also my model seek to replicate properties of Argentinean business cycle statistics.

As Hatchondo et al. (2010) demonstrated, the numerical solution strategy is all but a side issue for sovereign default models. They show quantitative results of Arellano and Aguiar and Gopinath to be tainted by inappropriate numerical solution methods. For example, while Aguiar and Gopinath claim that their model is capable of generating a substantial negative correlation between interest rate spread and trade balance, this result vanishes if accurate solution methods are used.

In face of these problems, I proceed in three steps: my model is simple enough to be solved analytically under the assumption that the government can perfectly commit itself to the repayment of its debt. As a first step, I study the quantitative properties of this model to understand to what extent pure consumption smoothing allows to generate realistic first and second moments of output, consumption, the trade balance and government debt. I find that such a model has only limited abilities to generate moments in line with Argentinean data. The main shortcoming of this model is that either, depending on the parametrization, the equilibrium level of debt much too large (private consumption is almost wiped out) or debt follows a random walk in output innovations, implying,

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1Real GDP taken from Maddison’s data, nominal government spending from Bureau of the Census (1975a, p. 1104, series 336) and inflation data taken from Bureau of the Census (1975b, p. 211, series 135). I detrended using the HP filter.

2See section 5.3 for details on sources a processing.


4Selective in 1956, and 1982/83, after the end of military dictatorship, and fully in 2001
amongst other things, a too small correlation of debt with output and consumption.

In a second step, I study the model with default under the simplifying assumption that the government can borrow only once. This allows to gain analytical results that give hints on the properties of the full model as well as starting values for the numerical solution of the full model. In the last step, I solve and simulate the full model. While the solution strategy of Hatchondo et al. is a substantial step forward, it does not deal with a seemingly intrinsic problem: one has to find four nonlinear policy functions to solve a basic sovereign default model. I show that the number of policy functions that have to be computed by numerical methods can be reduced to two: government’s policy function for debt issues and the default probability function as used by lenders. Apart from this, my algorithm works similar to the algorithm used by Aguiar and Gopinath but uses a continuous state space. The advantage of my approach of approximating only two functions comes to the price that the computation of the function values of the other two functions is time consuming, i.e. my algorithm is slow compared to the reported performance of the algorithms of Hatchondo et al.

I find that the possibility of a sovereign default heals many quantitative shortcomings of pure consumption smoothing and brings many moments close to the ones in Argentinean data. However, sovereign defaults will never occur if the time preferences of lenders and of the households, whose utility the government is maximizing, are very similar. Government debt may rise to levels well above 20% of GDP in peaks – but the government finds it always preferable to pay back its debt and the average level of government debt is close to zero. Therefore, a necessary condition for large average levels of government debt and a nonzero probability of a sovereign default is a substantial difference in time preferences. With lenders being more patient than households, the model can generate first and second moments broadly consistent with Argentinean data. In particular, my model can reproduce both the average level of Argentina’s debt to GDP ratio as well as its peaks.

2 The Model

Consider a small open economy inhabited by a representative household and a benevolent government. Let a period be a year and household’s preferences be given by

\[ U_0 = \sum_{t=0}^{\infty} \beta^t E_0[u(c_t)] \]

Where \( u_c > 0, u_{cc} < 0, 0 < \beta < 1 \) is the subjective discount factor and \( c_t \) is private consumption. For numerical simulations I use a CRRA utility function, augmented with a constant:

\[ u(c_t) = \frac{c_t^{1-\eta} - 1}{1 - \eta} + \eta_0 \]
This constant will be used to rule out negative levels of absolute utility that otherwise emerge for \( c_t < 1 \) given \( \eta > 1 \).

The government provides only indispensable services to the economy, like the enforcement of contracts, the securing of public order or the provision and maintenance of infrastructure. Therefore, very little government spending induces large frictions into the production of goods and services, zero spending implies the break down of public order with no consumable output left for households. The larger government spending, the larger the output due to a reduction of transaction cost, albeit this increase is tiny for high levels of government spending. This is modeled by assuming that the output of the economy is given as

\[
y_t = a_t k_t g_t^\gamma
\]

where \( a_t \) is the stochastically fluctuating productivity of the economy, \( k_t \) represents the effect of other factors of production and \( g_t \) is government spending. A very similar production function is used by some works of the growth literature, for example Barro (1990) and Barro and Sala-I-Martin (1992).

I assume that the law of motion for productivity is linear in logs:

\[
\ln(a_t) = \rho \ln(a_{t-1}) + \varepsilon_t \quad \text{where} \quad \varepsilon_t \sim N(0, \sigma^2)
\]

The government finances its expenditures with a proportional income tax \( \tau_t \) whose rate can be adjusted in every period. This implies that government’s budget constraint is \( \tau_t a_t k_t g_t^\gamma = g_t \) and household’s budget constraint is \( c_t = y_t(1 - \tau_t) \). The government is benevolent and thus seeks to maximize household’s utility by setting \( g_t \) optimally. In the setting of a closed economy, the optimal policy for \( g_t \) can be found by replacing \( \tau_t \) in households’ budget constraint, yielding the trivial optimization problem:

\[
\max_{g_t} \quad c_t = a_t k_t g_t^\gamma - g_t \quad \Rightarrow \quad g_t^* = \left( \gamma a_t \right)^{\frac{1}{\gamma - 1}}
\]

Inserting \( g_t^* \) in government’s budget constraint yields an optimal tax rate of \( \tau_t^* = \gamma \).

### 3 Calibration

Before extending the model, I shall discuss plausible parameter values for the setting just described. Because we just saw that \( \tau_t^* = \gamma \) in the closed economy, the average relation of tax revenue to GDP of an economy whose trade balance is zero on average allows to calibrate \( \gamma \). For Argentina, the trade balance as share of GDP was -0.7% on average, with a standard deviation of 2.8%. Argentina’s tax share of GDP is 14.3%\(^5\), therefore I calibrate \( \gamma = 0.143 \). This is more than twice the size of the equivalent number for the

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\(^5\)Tax share computed for 1993–2011 using data from MECON. See section 5.3 for details on data and processing.
early United States, for which Wallis (2000) reports that combined tax revenues of state and federal government were between 4 and 8.4% of GNP with an average of 5.9% for 1840-1900. However, the number is still much smaller than for, e.g., the modern United States.

The parameters $\rho$ and $\sigma$ must be estimated. In Appendix A I show that one can rewrite the model to obtain

$$\ln(y_t) - \ln(g_t) = -(1 - \rho) \ln(\gamma) + \rho(\ln(y_{t-1}) - \ln(g_{t-1}))$$

Estimating this equation with Argentinean data yields $\hat{\rho} = 0.625$. To estimate $\sigma$, I assume that the policy for $k_t$ follows the rule

$$\ln(k_t) = \psi_1 \ln(k_{t-1}) + \psi_2 \ln(a_t)$$

where $\rho \leq \psi_1 < 1$ and $0 < \psi_2 < 1$. With this additional assumption, the model implies that the data model of $\ln(a_t k_t)$ is an AR process of large order and the volatility of innovations of this AR process is equal to $\sigma$. Estimating the AR yields $\hat{\sigma} = 0.026$ (See appendix A for details).

Household’s subjective discount factor $\beta$ will not be calibrated directly. Rather, I shall define it in relation to the international risk free interest rate, which will be assumed to be identical to the inverse of international lenders’ subjective discount factor (to be discussed in the next section). I calibrate this interest rate as 1.55%, which is the average real interest rate of US treasury bonds with one year maturity.\footnote{Treasury yields taken from the Fed, CPI inflation taken from BLS, average computed for 1962–2012.} Thus, lenders’ subjective discount factor is calibrated as 0.985. I shall define this as $\xi \beta$, where $\beta$ is the subjective discount factor of households and study the effect of $\xi \neq 1$.

For the risk aversion, I choose $\eta = 2$, the standard value in the literature. For simplification, I will assume the factor endowment of the economy to be fixed in the following. First, because this makes the analytical solution of the model much simpler and second, because the first order condition of the model can be written in a way very similar to other models in the literature.

Summing up, the calibration is

| $\beta \xi$ | 0.985 | $\gamma$ | 0.143 | $\rho$ | 0.625 | $\sigma$ | 0.026 | $\eta$ | 2 | $k_t$ | 1 |

4 An Economy with Full Commitment

Now consider the economy in a setting where the government can borrow from risk neutral foreign lenders. Lenders can only lend / borrow once before leaving the model.
but have full information, including everything known to previous generation of lenders. They are assumed to be in perfect competition and can either invest in a risk free asset or lend to / borrow from the government. As pointed out by Cole and Kehoe (1995), the result of Bulow and Rogoff (1989) that there exists no equilibrium with positive debt if a government can save requires some technology that allows the government to save independently of its past behavior and the level of government debt. In my model (as in the works cited in the introduction) there exists only one set of lenders, meaning that the government can not do so. Furthermore, the government can not increase its revenue by investing in its own economy as in the model of Bulow and Rogoff. Therefore, the result of Bulow and Rogoff does not apply here.

In this section, the government is assumed to be able to commit itself to the repayment of its debt, making lending to it risk free. The government can borrow / save by selling / buying one period discount bonds. That is, if the government is in need of more of the consumption-capital good produced in its economy, it sells the pledge to transfer a unit of the good to lenders next period, who transfer $q$ units for this in the current period. $q$ is the price of a risk free benchmark asset, decomposed as $q = \beta \xi$ and assumed to be also the subjective discount factor of lenders. The interest rate is therefore $1\frac{1}{q}$. If $\xi \neq 1$, lenders preferences / the risk free interest rate differ from households preferences and we may expect persistent resource flows from / into the economy that yield a nonzero average level of government debt.

Government’s budget constraint is now

$$g_t = \tau_t a_t kg_t^\gamma + qB_t - B_{t-1}$$

The government is assumed to make sure that its spendings fulfill the intertemporal budget constraint:

$$\sum_{t=0}^{\infty} \frac{g_t}{\gamma q} + B_{-1} - \sum_{t=0}^{\infty} \frac{\tau_t a_t kg_t^\gamma}{\gamma q} = 0 \iff \sum_{t=0}^{\infty} \frac{qB_t - B_{t-1}}{\gamma q} + B_{-1} = 0$$

where $B_{-1}$ is the exogenously given initial level of government debt due to repay at period 0.

It is easy to see that the first order condition for the provision of $g_t$ does not depend on the level of debt or the bond price. Therefore, the government will provide $g_t$ by the very same rule as in the closed economy, which allows to rewrite government’s budget constraint as

$$\tau_t = \frac{(yka_t)^{\frac{1}{\gamma}}} {\gamma^{\frac{\gamma}{1-\gamma}} (ka_t)^{\frac{1}{\gamma}}} \leq 1$$

Note that I do allow for a negative tax rate. This has to be understood as transfer to households that offsets their formal tax payments. Now substitute out government spending
from households’ budget constraint to obtain:

\[ c_t = (\gamma k a_t)^{1/\gamma} (\gamma^{-1} - 1) + qB_t - B_{t-1} \]

The Lagrangian for government’s optimization problem is

\[ L = \sum_{t=0}^{\infty} E_0 \left[ \beta' u\left((\gamma k a_t)^{1/\gamma} (\gamma^{-1} - 1) + qB_t - B_{t-1}\right) - \lambda \left(\sum_{t=0}^{\infty} qB_t - B_{t-1}\right)^{1/q} \right] \]

The first order condition for government debt is

\[ \frac{\partial L}{\partial B_t} = E_0[\beta' qu'(c_t) - \lambda q^\gamma q - \beta^{t+1}u'(c_{t+1}) + \lambda q^{t+1}] = 0 \]

Note that the intertemporal budget constraint is not binding because of the constant interest rate \(1/q\). The first order condition yields the familiar condition

\[ \xi u'(c_t) = E_t[u'(c_{t+1})] \]

Using the functional form specified above, this can also be written as

\[ E_t[c_{t+1}] \frac{(\xi)^{1/\gamma}}{\Xi} = c_t \]

Thus, the model is simple enough to be solved analytically, the policy function is:

\[ B_t = \frac{(\gamma k)^{1/\gamma} (\gamma^{-1} - 1)(1 - \Xi)}{\Xi(q - 1)} - \tilde{a}_t \frac{(\gamma k)^{1/\gamma} (\gamma^{-1} - 1)}{\Xi} + \frac{1}{\Xi} B_{t-1} \]

where \(\tilde{a}_t \equiv \ln(a_t)\). For details on the computation, see Appendix B. A plausible initial value for \(B_t\) is zero or a number close to. This is because Argentina had very low debt to GDP ratios in the early 1970s, with a minimum of 6.5% in 1974.

The policy function reveals that the time path of debt depends very much on the relation of the risk free world interest rate and household’s preferences. If \(\xi > 1\), lenders are more willing to sacrifice present consumption for future consumption than the households in the economy. If \(\xi > 1\), \(\Xi > 1\) too, therefore the government will issue debt to allow for more present consumption, up to a point where output is barely sufficient to maintain debt service. As example, consider \(\beta = 0.95\), which translates to \(\xi = 1.037\). In this case, \(B_t\) will converge to an equilibrium level of about 42. Since equilibrium output is

As pointed out by Bohn (2007, p. 1840), the transversality condition (TVC) \(\lim_{n \to \infty} q^n E_t[B_{t+n}] = 0\) is the link between the intertemporal budget constraint (IBC) and the intratemporal one in an environment of constant interest rates. Therefore, one may either study the TVC or the IBC. On the effect of the constant interest rate on the TVC, Bohn writes: "The discounting in the transversality condition is exponential in \(n\). Exponential growth is known to dominate polynomial growth of any order. Hence the discount factor \(q\) will asymptotically dominate \(E_t[B_{t+n}]\) whenever debt is difference-stationary with arbitrary order of integration."
approximately 0.72, this is roughly 58 times GDP. However, to roll over this level of debt forever means that the government has to pay 0.62 units per period to foreign lenders and thus leaves a puny level consumption for the households. This implication makes the perfect commitment of the government hard to believe in since a default would benefit households rather tremendously in this setting. Now consider \( \xi = 1 \). This simplifies the policy function to

\[
B_t = -\tilde{a}_t (y k)^{\frac{1}{1-\gamma}} \frac{1}{1-\gamma} (y^{-1} - 1) + B_{t-1}
\]

Which is \( B_t = -0.7\tilde{a}_t + B_{t-1} \) for the calibration discussed above. In this case, the unconditional expected value of government debt is zero, but numerical simulations exhibit wide swings of government debt:

**Figure 1:** Numerical simulation of perfect commitment model with \( \xi = 1 \).

If \( \xi < 1 \), households in the economy are more willing to save than international lenders. In this case, the model is explosive in the sense that the economy will build up an ever increasing amount of savings which generate an ever increasing level of consumption for households.

## 5 A Government that can default

Now allow the government to choose whether to repay or default on its debt. To investigate this setting, I proceed in two steps. First, I study a simplified version of the model in which the government borrows only once. I gain some analytical as well as accurate quantitative results that serve as initial values for the numerical solution of the full model.

The general framework is similar for variants of the model. I assume that in case of a default, the government is forced to repay the share \( 0 \leq \delta \leq 1 \) of the bond nonetheless.
For simplification, I assume this share to be fixed, viz there is no negotiation on δ (see Benjamin and Wright (2009) and Yue (2010) for models with endogenous δ). For numerical simulations I calibrate δ = 0.6, which is the average of about ninety defaults that occurred 1979–2005 as reported by Benjamin and Wright. Such an incomplete haircut has a very simple, yet important implication: Lenders can only expect a government in default to repay δBt, if yt+1 ≥ δBt, which implies an upper bound of Bt ≤ E[yt+1/δ].

Denote government’s default decision with zt ∈ {0, 1}, where zt = 1 means that the government repays and zt = 0 means that the government is in default, it is then excluded from the capital market. A government in default may regain access to the capital market with probability θ. A government in default faces output effects that are modeled by an extension of the production function:

\[ y_t = a_t k z_t + v \]

Following Aguiar and Gopinath (2006) I calibrate \( \frac{v}{1 + v} \equiv \kappa = 0.98 \), viz the economy looses two percent of output if the government is in default. The output effect introduces a link between government spending and zt. To make this as simple as possible, I assume that the government first decides on zt and then on gt, making zt deterministic in government’s spending policy.

In line with the literature, I assume output effects of a default to last as long as the capital market exclusion for the full model. Gelos et al. (2011) report an average duration of capital market exclusion of about three years, which translates to θ = 0.33.

In each period, the order of events is the following:

1. The realization of \( a_t \) is revealed to the players.
2. Given \( z_{t-1} = 1 \), the government decides on \( z_t \). If \( z_{t-1} = 0 \), nature draws whether renewed access to the capital market is granted. If access is granted, \( z_t = 1 \), if not \( z_t = 0 \)
3. Given \( z_t = 1 \), the government issues new bonds / offers savings to lenders. After an agreement with lenders, the equilibrium bond price is \( q_t \).
4. The government spends \( g_t \), the economy produces, households consume.

Lenders are identical, risk neutral and two period lived. Each lender enters the model with a large wealth \( W \) and can finance next period’s consumption either by trading a risk free at the international capital market or by buying bonds from / selling bonds to the government. This implies that lender’s optimization problem is

\[
\max_{c_t, \alpha} \beta^{\xi} \left( \alpha \left( \frac{W - c_t}{q_t} P(z_{t+1} = 1) + (1 - P(z_{t+1} = 1)) \delta \frac{W - c_t}{q_t} \right) + (1 - \alpha) \frac{W - c_t}{\beta^{\xi}} \right)
\]

The first order condition for both \( c_t \) and \( \alpha \) is:

\[
q_t = \beta^{\xi} \left( P(z_{t+1} = 1) + (1 - P(z_{t+1} = 1)) \delta \right)
\] (1)
5.1 Exact results for a special case

Simplifying assumption. The economy is closed after the next period. That is, a government can lend in period 0 and might repay in period 1, but is excluded from the capital market from period 1 on while the output effect of a default lasts $\omega = 1/\theta = 3$ periods.

By this assumption, consumption in period 1 is

$$c_1(z_1) = \begin{cases} (\gamma \kappa a_1) \frac{1}{1-\gamma} (\gamma^{-1} - 1) - B_0 & \text{if } z_1 = 1 \\ \kappa (\gamma \kappa a_1) \frac{1}{1-\gamma} (\gamma^{-1} - 1) - \delta B_0 & \text{if } z_1 = 0 \end{cases}$$

The government will choose to repay the bond iff the expected utility of doing so is greater than the utility of not doing so, viz if:

$$\Delta U_1 = u(c_1(z_1 = 1)) - u(c_1(z_1 = 0)) + E_t \left[ \sum_{j_1 = 1}^{\omega} \beta^{j_1} u(c_{1+j_1}(z_1 = 1)) \right] \ldots$$

$$\ldots + \sum_{j_1 = \omega + 1}^{\infty} \beta^{j_1} u(c_{1+j_1}(z_1 = 1)) - \sum_{j_1 = 1}^{\omega} \beta^{j_1} u(c_{1+j_1}(z_1 = 0)) - \sum_{j_2 = \omega + 1}^{\infty} \beta^{j_2} u(c_{1+j_2}) > 0$$

Utility after period $\omega$ cancels out, denote:

$$\Delta U_1 = u(c_1(z_1 = 1)) - u(c_1(z_1 = 0)) + E_t \left[ \sum_{j_1 = 1}^{\omega} \beta^{j_1} u(c_{1+j_1}(z_1 = 1)) \right] > 0$$

(2)

Because the economy is closed in either path after period 1, one can rewrite each element in the sum of $\Delta V_1$ as

$$(1 - \kappa^{1-\eta}) \left( \frac{c_{1+j}}{1-\eta} \right)^{1-\eta}$$

Replace consumption with its definition and write out $\Delta V_1$:

$$\Delta V_1 = \sum_{j_1 = 1}^{\omega} \beta^{j_1} \frac{1 - \kappa^{1-\eta}}{1-\eta} (\gamma k) \frac{1}{1-\gamma} (\gamma^{-1} - 1)^{1-\eta} \exp \left( \frac{1 - \eta}{1 - \gamma} \tilde{a}_{1+j} \right)$$

Since $E[\tilde{a}_t] = 0$, approximate:

$$\Delta V_1 \approx \sum_{j_1 = 1}^{\omega} \beta^{j_1} \tilde{a}_1 \left( \frac{1 - \eta}{1 - \gamma} \tilde{a}_{1+j} + 1 \right)$$

Every agent in the model can forecast $c_{t+j}$ using information from period $t$ as:

$$E_t[c_{t+j}] = (\gamma k) \frac{1}{1-\gamma} (\gamma^{-1} - 1) \left( 1 + \frac{\rho^j}{1-\gamma} \tilde{a}_t \right)$$
The government uses
\[ E_1[\Delta V_1] = \sum_{j=1}^{\omega} \beta^j \sigma_1 \left( \frac{\rho^j (1 - \eta) \tilde{a}_1 + 1}{1 - \gamma} \right) \]
while lenders in period 0 design the final loan contract using
\[ E_0[\Delta V_1] = \sum_{i=1}^{\omega} \beta^i \sigma_1 \left( \frac{\rho^{i+1} (1 - \eta) \tilde{a}_0 + 1}{1 - \gamma} \right) \]

Lenders in period 0 know that \( P(\Delta U_1 < 0) = P(\text{default}) \). Obviously, this probability is a function of both \( \tilde{a}_0 \) and \( B_0 \), but due to the nonlinear utility function, one cannot isolate either variable from \( \Delta U_1 \) and therewith \( \Delta U_1 \). However, one can easily obtain a "safe" level of debt, that is: a level of debt where the probability of default is close to zero. This is the case for loan contracts that imply \( E_0[c_1 \mid z_1 = 1] \geq E_0[c_0 \mid z_1 = 0] \). If this is the case, the difference between the felicity of repayment and default \( \Delta U_1 \) is positive and since \( \Delta V_1 \) is always positive, \( \Delta U_1 > 0 \) for sure. To make the government default nonetheless, \(-\epsilon_1\) must be large enough to not only make \( \Delta U_1 < 0 \) but also compensate for \( \Delta V_1 \). The latter can be understood as the utility gain of having \( 1 - \kappa \) (2% for my calibration) more consumption for the next \( \omega \) periods. Given \( \sigma \) is assumed to be quite small, such a large \(-\epsilon_1\) is very unlikely. This "safe" level of debt is readily computed as
\[ B_0 \leq E_0[y_1 - g_1] \frac{1 - \kappa}{1 - \delta} \]
which is 4.3% of equilibrium output for the calibration discussed above. It is also easy to compute a level of debt for which the probability of a default is 100% because the level of debt exceeds the repayment capabilities of the government. Given \( \kappa > \delta \), this is \( B_0 \geq E_0[y_1 - g_1] \), which is about 86% of equilibrium output.

For \( E_0[y_1 - g_1] \in (\frac{1 - \kappa}{1 - \delta}, 1) \), the default probability used by lenders is some function \( d_0(B_0, \tilde{a}_0) \).

Proposition 1. In the area where the default probability is neither zero nor 100%, it is true for the default probability function \( d(B_0, \tilde{a}_0) \), given \( \kappa > \delta \), that
\[ \frac{\partial d(\cdot)}{\partial B_0} > 0 \quad \text{and} \quad \frac{\partial d(\cdot)}{\partial \tilde{a}_0} < 0 \]

Proof. See appendix C.1. 

One can interpret \( \frac{\partial d(\cdot)}{\partial \tilde{a}_0} < 0 \) as negative relation of current output and next period’s default risk, yielding a negative correlation of the interest rate \( 1/\rho_0 \) and current output.

Remark: For the next proof and the solution of the full model, note that if lenders are rational, then \( d_t = P(z_{t+1} = 0 \mid \text{information in } t) \). If \( d_t \neq P(z_{t+1} = 0 \mid \text{information in } t) \), lenders would make systematic errors, which is incompatible with rationality under symmetric information.
Proposition 2. The equilibrium loan size $B^*_0$ implies $d(B^*_0, \tilde{a}_0) \approx 0$ if

$$B^{-1} < \beta E_0[y_1 - g_1] (1 + \delta \xi)^{1 - \kappa} \frac{1}{1 - \delta}$$

Proof. See appendix C.2

This “safe” level of inherited debt is about 6.8% of equilibrium output.

Thus, both for $B^{-1}$ and $B_0$ there exists a “safe” level of debt. Under the simplifying assumption, the government has meagre benefits from repaying its debt. This has an important implication for the full model: In the full model, repaying debt implies continued access to the capital market and thus greater benefits from repayment. If the government in the full model is not very “aggressive” in taking on new debt, viz is reluctant to increase debt much in the face of a negative output innovation, it is entirely possible that the full model will exhibit an interest rate spread of zero for many periods.

To gain more intuition on the shape of $d(B_0, \tilde{a}_0)$, I perform a numerical simulation. I set $B^{-1} = 0$ and create a grid for the remaining state variable $\tilde{a}_0$ and the policy variable $B_0$. For each point in the grid, I generate 1000 realizations of $\epsilon_1$ and compute the probability of a default for each point in the grid using condition (2). The result is depicted in figure 2.

Figure 2: $d(B_0, \tilde{a}_0)$ for the simplified model

The simulation confirms the results of proposition 1. It indicates that $d(B_0, \tilde{a}_0)$ can be satisfactorily described by a variant of the Logit model:

$$d(B_0, \tilde{a}_0) \approx \frac{\exp \left( m_1(\tilde{a}_0) + m_2(\tilde{a}_0)B_0 \right)}{1 + \exp \left( m_1(\tilde{a}_0) + m_2(\tilde{a}_0)B_0 \right)}$$

(3)
5.2 The general model

A government with a per se unlimited access to the capital market is confronted with a demanding optimization problem. In essence, this is because an evaluation whether a default today is beneficial requires an evaluation of the option value of a default in later periods and of the possibility that the government might return to the capital market after a default. Begin with the common way to define government’s decision problem as pioneered in the early versions of Arellano (2008). The value function of a government with access to the capital market can be defined as

\[ V_1(\tilde{a}_t, B_{t-1}) = \max_{B_t} \left\{ u(y_t - g_t + q_tB_t - B_{t-1}) + \beta E_t \left[ V_1(\tilde{a}_{t+1}, B_t) \right] \right\} \]  

(4)

where \( V(\cdot) \) describes the value given the option to default:

\[ V(\tilde{a}_t, B_{t-1}) = \max_{z_t \in \{0,1\}} \left\{ z_tV_1(\tilde{a}_t, B_{t-1}) + (1 - z_t)V_0(\tilde{a}_t, B_{t-1}, 0) \right\} \]

and where \( V_0(\cdot) \) is the value function of a government under default. Since \( \delta \) might be larger than 0, one has to distinguish between the period where the government decides to set \( z_t = 0 \) and periods after the default decision:

\[ V_0(\tilde{a}_t, B_{t-1}) = \begin{cases} 
    u(\kappa(y_t - g_t) - \delta B_{t-1}) + \beta E_t \left[ \theta V_1(\tilde{a}_{t+1}, 0) + (1 - \theta)V_0(\tilde{a}_{t+1}, 0) \right] & \text{if } z_t = 0 \\
    u(\kappa(y_t - g_t)) + \beta E_t \left[ \theta V_1(\tilde{a}_{t+1}, 0) + (1 - \theta)V_0(\tilde{a}_{t+1}, 0) \right] & \text{if } z_{t-j} = 0 
\end{cases} \]  

(5)

Due to (1), the policy function for the bond price is a linear transformation of the default probability function \( d(\tilde{a}_t, B_t) = P(\arg\max V(\tilde{a}_{t+1}, B_t) = 0) \).

These functions allow to define:

**Definition 1.** The recursive equilibrium of the economy is defined by (i) a policy function for government’s bond issue \( B_t = B(\cdot) \), (ii) the value functions \( V(\cdot), V_1(\cdot) \) and \( V_0(\cdot) \), (iii) the policy function for the bond price \( q_t = q(\cdot) \) charged by lenders, (iv) government’s optimal spending policy \( g_t(\cdot) \) and (v) households’ consumption policy. These functions have the properties:

- (ii) contains (v) and (iv).
- (i) maximizes (ii) given (iii) subject to the resource constraint of the economy.
- (iii) maximizes lender’s utility given (i) and their resource constraint.
- (v) maximizes household’s consumption and satisfies their resource constraint.

Note that, because (iv) is very easy to define and households have zero degrees of freedom, (iv) and (v) are trivial to characterize and integrated in (ii) in the following.
The possibility that the government might default now or in any future period opens an infinite tree of possibilities that makes a quantitative implementation of the model demanding. However, this task can be eased by an implicit assumption of the model which, to my best knowledge, has not yet been exploited in the literature: there is no asymmetric information and both players are rational. This means that in equilibrium the default probability \( d_t = d(\tilde{a}_t, B_t) \) used by lenders in period \( t \) must be exactly identical to \( P(\arg\max V(\tilde{a}_{t+1}, B_t) = 0) \) as expected by the government in period \( t \) because both use the same information set. This allows to rewrite (4) as

\[
V_1(\tilde{a}_t, B_{t-1}) = \max_{\tilde{a}_t} \left\{ u(y_t - g_t + q_t B_t - B_{t-1}) + \beta \left[ (1 - d_t) E_t[V_1(\tilde{a}_{t+1}, B_t)] \right] \cdot \cdots + d_t E_t[V_0(\tilde{a}_{t+1}, B_t, 0)] \right\}
\]

One can extend \( V_0(\cdot) \) and \( V_1(\cdot) \) even further. The resulting function is messy but a closer investigation reveals that it is simply a sum of the discounted different possible future felicities (the branches of the tree of possibilities), weighted by simple functions of \( E_t[d_{t+1}] \) and \( \theta \). Therefore, the computation of \( V_1(\tilde{a}_t, B_{t-1}) \) and \( V_0(\tilde{a}_t, B_{t-1}) \) is straightforward: It is easy to compute the structure of the tree of possibilities, viz the \( 2^l \) paths that must be considered by a rational government that looks \( l \) periods ahead. Each path represents a possible sequence of events whose expected utility can be computed using numerical integration, given one knows \( d(\cdot) \) and the policy function for bond issues. This means that all that is needed for a solution is these two functions.

It is also easy to see that \( d(\cdot) \) must have a shape similar to (3) because the default probability is monotonous in government debt.\(^8\) To allow for nonlinearities in \( \tilde{a}_t \), I approximate \( m_1(\cdot) \) and \( m_2(\cdot) \) with 6th order polynomials. In the perfect commitment model, \( B(\cdot) \) is linear. For the model without perfect commitment, I allow for nonlinearities by choosing a third order polynomial. To solve the model, I use an algorithm that works as follows:

1. Initialization: simulate simplified model to gain an initial value for \( d(\cdot) \), use policy function from perfect commitment model as initial value for \( B(\cdot) \).

2. Find \( d(\cdot) \) given \( B(\cdot) \).
   
   (a) Let lenders use \( d(\cdot) \) to offer contracts to the government, use \( d(\cdot) \) and \( B(\cdot) \) to compute the values of \( V_1(\cdot) \) and \( V_0(\cdot) \), which provide the actual \( P(z_{t+1} = 1) \) that emerge in the grid.

---

\(^8\)The reason is the same as in Eaton and Gersovitz (1981, theorem 1). It might be possible that the government chooses to repay some large loans with probability one, given the bond price is moderately high, while it defaults on smaller and larger loans with probability one. However, such a region can only be reached by passing regions where \( d_t = 1 \). The government can not commit to keep debt in the region, therefore lenders must use \( d_t > 0 \) also for debt in such a region, which in turn makes a rolling over of such a large amount of debt unattractive for governments and destroys the incentive for sure repayment.
(b) Estimate \(d(\cdot)\) using these actual default probabilities.

(c) Check if the difference between the actual \(P(z_{t+1} = 0)\) and \(P(z_{t+1} = 0 \mid t) = d(\cdot)\) decreased relative to the last iteration. If yes, mix new estimate with old parameters to avoid oscillations and return to (a). If the difference is below threshold value go to 3. If the difference stopped decreasing but is above the threshold, use MATLAB’s fminsearch routine to minimize the difference and then go to 3.

3. Given \(d(\cdot), \text{search } B(\cdot).\) The first order condition for \(B_t\) is \(\frac{\partial V(\cdot)}{\partial B_t}.\) It consists of a ”\(t\) and \(t + 1\) part” that can easily be expressed analytically and of a part that describes the effect on future utility, which is rather messy and has to be computed numerically. The first order condition is used to gain the correct \(B_t.\) Given this value, I compute the policy function by minimizing the squared difference between the correct \(B_t\) and \(B_t\) as implied by \(B(\cdot).\)

4. Return to 2. until a convergence of the parameters in \(B(\cdot).\)

For details on the model solution see Appendix (D).

5.3 Benchmark Data & Results

In contrast to other studies, I compare and calibrate the model with annual data from 1960 to 2011. In my opinion, the disadvantages of a coarser sample frequency are outweighed by the fact that such a data set also covers the period before Argentina’s default in late 1982. My main source for data is the World Bank. Where World Bank data is incomplete, I use data from the IMF’s world economic outlook database. To compute government spending for 1968 – 2011, I used data published by Argentina’s ministry for Economics and Finance MECON. For the interest rate spread \(S,\) I use J.P. Morgan’s Emerging Market Bond Index spread as provided by the World Bank.\(^9\) Note that J.P. Morgan provides historic values of the index only back to 1993 (cf. Cavanagh and Long, 1999). Government’s primary spending \(g,\) private consumption \(c\) and GDP \(y\) were de-trended using the HP filter. \(TB\) stands for the relation of the current account balance to GDP and \(D\) for Debt to GDP. All displayed correlations are computed for the respective largest simultaneous availability of the time series.

\(^9\)See (10) in the proof of proposition C.2 for an explicit expression under the simplifying assumption from the last section.

\(^{10}\)As pointed out by Ludwig (2013), the use of EMBI spreads might yield a distorted picture of spreads – simply because the EMBI was not constructed to give the best possible picture of interest rate spreads. Therefore, I also computed the correlations with the spread index proposed in Ludwig (2013) but find only tiny differences.
Table 1: Business Cycle Statistics

<table>
<thead>
<tr>
<th></th>
<th>Correlations</th>
<th>Volatilities</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>y  c  g  S  TB</td>
<td>σ(y): 5.7%</td>
</tr>
<tr>
<td>c</td>
<td>0.85</td>
<td>σ(c): 7.7%</td>
</tr>
<tr>
<td>g</td>
<td>0.66 0.57</td>
<td>σ(g): 11.6%</td>
</tr>
<tr>
<td>S</td>
<td>-0.89 -0.80 -0.53</td>
<td>σ(S): 18.0%</td>
</tr>
<tr>
<td>TB</td>
<td>-0.67 -0.74 -0.58 0.67</td>
<td>σ(TB): 2.8%</td>
</tr>
<tr>
<td>D</td>
<td>-0.67 -0.56 -0.49 0.96 0.39</td>
<td>Mean(D): 36.3% Max(D): 121.4%</td>
</tr>
</tbody>
</table>

These numbers are in line with numbers computed by Neumeyer and Perri (2005) and in works cited above, with one exception: the correlation of TB and S has the opposite sign. This is probably because Neumeyer and Perri (2005) and other authors use non-public data on a number of bonds issued in the early 1980s to augment EMBI spread data for the time before 1993. However, for 1993 – 2011, the correlation between the trade balance and the interest rate spread is undeniably positive:

**Figure 3:** Trade balance to GDP ratio and interest rate spread for Argentina

![Figure 3: Trade balance to GDP ratio and interest rate spread for Argentina](image)

Monthly data, seasonally adjusted; source: World Bank and author’s computations. GDP per month is computed as average of the respective quarter.

To disentangle the effect of the basic modeling approach of explaining government debt by consumption smoothing and the effect of the possibility of a sovereign default in this framework, I shall begin the investigation into the empirical qualities of the model by investigating the variant with perfect commitment. I call this model 1. In table 2, model 1a is model 1 with ξ = 1 while model 1b is the model with ξ = 1.037. This
difference in preferences is conservative compared to the differences implicitly used in the literature, yet table 2 reveals that even this small difference has huge and generally unfavorable effects in the model with perfect commitment. I am therefore skeptical whether it is sensible to assume much larger differences in preferences.

**Table 2: Business Cycle Statistics**

*Perfect Commitment Model*

<table>
<thead>
<tr>
<th>Model 1a</th>
<th>y</th>
<th>c</th>
<th>g</th>
<th>TB</th>
</tr>
</thead>
<tbody>
<tr>
<td>c</td>
<td>0.03</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>g</td>
<td>1.00</td>
<td>0.03</td>
<td></td>
<td></td>
</tr>
<tr>
<td>TB</td>
<td>0.93</td>
<td>-0.32</td>
<td>0.93</td>
<td></td>
</tr>
<tr>
<td>D</td>
<td>-0.02</td>
<td>-0.99</td>
<td>-0.02</td>
<td>0.32</td>
</tr>
</tbody>
</table>

\[
\sigma(y) = 3.9\% \quad \sigma(c) = 0.15\%
\]
\[
\sigma(g) = 3.9\% \quad \sigma(TB) = 3.6\%
\]
\[
\sigma(D) = 83.6\%
\]

Mean(D)=-0.3\% Max(D)=174.6\%

<table>
<thead>
<tr>
<th>Model 1b</th>
<th>y</th>
<th>c</th>
<th>g</th>
<th>TB</th>
</tr>
</thead>
<tbody>
<tr>
<td>c</td>
<td>0.01</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>g</td>
<td>1.00</td>
<td>0.00</td>
<td></td>
<td></td>
</tr>
<tr>
<td>TB</td>
<td>0.00</td>
<td>-1.00</td>
<td>0.00</td>
<td></td>
</tr>
<tr>
<td>D</td>
<td>-0.24</td>
<td>-0.97</td>
<td>-0.24</td>
<td>0.97</td>
</tr>
</tbody>
</table>

\[
\sigma(y) = 3.9\% \quad \sigma(c) \geq 200\%
\]
\[
\sigma(g) = 3.9\% \quad \sigma(TB) = 29.8\%
\]
\[
\sigma(D) > 200\%
\]

Mean(D) > 200\% Max(D)>200%

The numbers were computed by simulating the model for 1000 years (starting from zero debt), computing statistics, repeating this 100 times and taking averages. Note that, except for debt, model 1a is stationary, while statistics for model 1b also cover the instationary convergence of the model to a low consumption regime (which takes roughly 80 years). For this reason, many statistics of this model are not meaningful. Further note that since the debt policy function implies that government debt is a random walk in \( \tilde{a} \), for model 1a, its debt volatility can be made arbitrary large by increasing the simulation horizon.

In the model, government debt is the only reason for in- and outflows of resources, therefore the trade balance is simply \(-(q_{B} - B_{-1})\). Table 2 exhibits a counterfactual high positive correlation of \( TB \) and \( y \) in model 1a. This is a result of consumption smoothing: if a positive output shock occurs, the government lowers the level of debt, which means an outflow of resources and vice versa. In model 1b, the trade balance is permanently negative during convergence into the low consumption regime and then permanently positive due to the infinite rolling over of debt. Both weakens the response of the trade balance to output shocks.

Comparing table 1 and 2 one finds that the ability of model 1a to replicate properties of Argentina’s business cycle is very limited. The reason for its unfavorable properties 11The authors usually calibrate the risk free interest rate to values around 1.5\% and sometime use \( \beta \) below 0.9.
this can be found by inspecting the policy function: nearly \( \frac{3}{4} \) of a technology shock is "transmitted" into debt while debt from previous periods is rolled over unchanged, producing a strong positive correlation of \( TB \) and \( y \) and a very weak correlation of \( D \) and \( y \). Model 1b is at the other side of the spectrum, with relatively realistic correlations of \( D \) with other variables but almost no realistic volatilities. In both variants, government’s ability to commit perfectly to the repayment of its debt has the counterfactual implication that smoothing household’s consumption works too perfect, the correlation of consumption with output almost vanishes.

The comparison of correlations between \( TB \) and \( D \) highlights a simple, yet surprising implication of models where government debt is the sole reason for resource in- and outflows: Even if government debt is positive and rising, the trade balance might be in surplus. To see this note that if the government is indebted, the economy exhibits a trade deficit if

\[-(q_t B_t - B_{t-1}) < 0 \iff \frac{1}{q_t} \frac{B_t}{B_{t-1}} > 1 \]  

(6)

For the perfect commitment model this means that the growth rate of debt must exceed \( \frac{1}{\beta \xi} - 1 \) to make the trade balance negative. Table 2 exhibits that this was not the case in model 1b and just very few times in model 1a, yielding a smaller but still positive correlation. In contrast, the correlation for Argentina is negative. In terms of the model, this is consistent with Argentina’s substantial average growth rate of debt of about 7%.

Consider now the model with default and call this model 2. Again, model 2a is the model with \( \xi = 1 \) and model 2b the model with \( \xi = 1.037 \). Table 3 exhibits its properties. The numbers in the table were generated similarly to the numbers in table 2.\(^{12}\) Table 3 indicates that the incorporation of a default tends to bring the business cycle statistics of the model much closer to those of Argentina. However, a major shortcoming exposed in model 1 is not healed: with the current interim version of the solution algorithm, the model can not generate a negative correlation between \( y \) and \( TB \), which, as mentioned above, is plausible for the model but counterfactual for Argentina. I suspect that when the algorithm is completed\(^{13}\), the result might be similar to the results of Hatchondo et al. (2010) who reproduce the negative correlation with Arellano’s and Aguiar and Gopinath’s model, i.e. in a very similar framework.

The correlation of \( D \) and \( y \) in both variants of the model is roughly in line with Argentinean data, which is a result of the smaller weight of debt from \( t - 1 \) in the policy function. In line with model 1, consumption smoothing works still "too perfect" to reproduce the large positive correlation observed in Argentina.

\(^{12}\)Note that because the algorithm presented in the last section is still work in progress, the displayed results have been generated by an interim version of the algorithm, with a linear policy function, looser convergence criteria and less foresight.

\(^{13}\)At the moment (June 28, 2013), the algorithm is too slow to make the use of a nonlinear policy function for \( B_t \) feasible. Therefore, I am rewriting the code to use MATLAB’s parallel computing toolbox and to make it general enough to be suitable for any sovereign default model with two state variables.
The model does not generate a negative correlation between $S$ and $TB$, which is also in line with results of Hatchondo et al. (2010). Since I measure a positive correlation for Argentina in 1993–2011, while Neumeyer and Perri (2005) and others measure a negative correlation for 1983–2001, it is not clear whether this is a advantage or disadvantage of the model. However, this property will not vanish with the complete version of the algorithm since it is a result of the modeling approach: An increase of $S$ means a decrease of $q_t$. This in turn means that the trade balance $- (q_tB_t - B_{t-1})$ c.p. improves due to the larger resource outflow. A negative correlation would require a resource inflow when $q_t$ drops, that is: a drastic increase of $B_t$ that counterbalances the decrease in $q_t$, which can not be a rational strategy for the government.\footnote{The reasoning for this is essentially the same as for proposition 2 of Arellano (2008).} The model also generates a volatility of $S$ and correlations of $S$ with other variables that are substantially smaller than their empirical counterparts. The reason for this is that the spread is zero quite often because the economy is not close to regions in the state space where the default risk has a substantial positive value. This possibility has already been indicated by proposition 2 where I showed that, even in a setting where the government benefits little from repaying its debt, there exist positive $B_t$ and $B_{t-1}$ that imply $d_t \approx 0$.

A striking difference to results in the literature is that my model generates quite large debt to GDP ratios with a moderate difference in preferences. The reason for this is that, except for Yue (2010), the literature assumes $\delta = 0$. A large $\delta$ deprives a government from a great chunk of the benefits of a default and thus increases government’s incentives to repay, allowing for higher equilibrium levels of debt. Unfortunately, this very ability of the government to credibly commit to repay also relatively high levels of debt is also

The model also generates a volatility of $S$ and correlations of $S$ with other variables that are substantially smaller than their empirical counterparts. The reason for this is that, except for Yue (2010), the literature assumes $\delta = 0$. A large $\delta$ deprives a government from a great chunk of the benefits of a default and thus increases government’s incentives to repay, allowing for higher equilibrium levels of debt. Unfortunately, this very ability of the government to credibly commit to repay also relatively high levels of debt is also

\begin{table}[h]
\centering
\caption{Business Cycle Statistics}
\label{tab:business-cycle-stats}
\begin{tabular}{llllll}
\hline
 & $y$ & $c$ & $g$ & $S$ & $TB$ \\
\hline
$c$ & 0.37 & & & & \\
$g$ & 1.00 & 0.37 & & & \\
$S$ & 0.00 & 0.00 & 0.00 & & \\
$TB$ & 0.84 & -0.21 & 0.84 & 0.00 & \\
$D$ & -0.61 & -0.96 & -0.61 & 0.00 & -0.08 \\
\hline
\end{tabular}
\end{table}
the reason why consumption smoothing works “too good”. Arellano (2008) assumes \( \delta = 0 \), yielding low average levels of debt but also a high correlation between \( c \) and \( y \).

6 Conclusion

I presented a simple neoclassical approach of modeling a government’s role in the economy, which improves the microeconomic underpinning of stochastic general equilibrium models for sovereign defaults on side of the government. In terms of reproducing Argentinean business cycle statistics, the approach seems to work satisfactory well, albeit results of the final algorithm, to be published later this year, must ultimately show the model’s worth. I also highlighted the importance of preference differences for the possibility of sovereign defaults in a neoclassical framework. This difference is the sole reason for positive interest rate spreads / the possibility of a sovereign default in equilibrium, therefore it is critical and deserves empirical scrutiny.

The computation of benchmark numbers raises a surprising question: is the model’s inability to reproduce a negative correlation between spread and trade balance really a disadvantage in case of Argentina? Seemingly, whether the empirical correlation is positive or negative depends on the time window investigated. For 1993–2011, all correlations of the spread to other variables generated by the model have correct signs but tend to be too small in size. This property is shared by other models in the literature and Arellano (2008) suggested from the early versions of her paper on that the solution to this problem might lie in risk averse lenders. Arellano’s idea was put to practice by Lizarazo (2013) and indeed, she is able to generate realistic spread volatilities and larger, yet still not realistically large, correlations of the spread with other variables. Therefore, I suspect a second driver of spreads to be in place: liquidity effects. Even though the EMBI index is designed to capture only relatively liquid bonds, the liquidity of Argentinean government bonds – and even more bonds of smaller emerging market sovereigns – is nonetheless meagre compared to bonds of many rich world countries. The incorporation of such liquidity effects, as well as the introduction of a nominal model with flexible exchange rates, has to be on the agenda of future research.

References


A Estimating ρ and σ

To estimate these parameters, one needs to rewrite the model in terms of variables for which data is available (that is: not in terms of $k_t$). This is easy for $\rho$. Begin with the optimal level of government spending:

$$\ln(g_t) = \frac{1}{1-\gamma} \ln(\gamma a_t)$$

Combining this with the law of motion for technology yields a law of motion for government spending:

$$\ln(g_t) = \frac{1}{1-\gamma} ( \ln(\gamma k_t) - \rho \ln(\gamma k_{t-1}) ) + \rho \ln(g_{t-1}) + \frac{1}{1-\gamma} \epsilon_t$$

Similarly, one can combine the production function with the optimal level of government spending:

$$\ln(y_t) = \ln(a_t k_t) + \frac{\gamma}{1-\gamma} \ln(\gamma a_t k_t) = \frac{1}{1-\gamma} \ln(\gamma a_t k_t) - \ln(\gamma)$$

and insert the law of motion for technology to obtain a law of motion for output

$$\ln(y_t) = -(1-\rho) \ln(\gamma) + \frac{1}{1-\gamma} ( \ln(\gamma k_t) - \rho \ln(\gamma k_{t-1}) ) + \rho \ln(y_{t-1}) + \frac{1}{1-\gamma} \epsilon_t$$
However, both laws of motion contain the catch call variable for other production factors and are thus not useful for estimation. Yet, replacing the second term with the law of motion for government spending in the last expression yields a simple law of motion for the variable $x_t = \ln(y_t) - \ln(g_t)$:

$$\ln(y_t) - \ln(g_t) = -(1 - \rho) \ln(\gamma) + \rho(\ln(y_{t-1}) - \ln(g_{t-1}))$$  \hspace{1cm} (7)

This equation can be estimated. Since the model omits investments, I use real GDP minus investments as $y_t$ for estimation\textsuperscript{15}. Both $\ln(p_y q\prime)$ and $\ln(p_g q)$ were detrended using the HP filter. Estimating (7) yields:

$$x_t = 0.625 x_{t-1}$$  \hspace{1cm} (2.89)

Be sure to note that the estimated standard error of innovations of this regression has nothing to do with $\sigma$. To estimate $\sigma$, note that one can easily compute $a_t k_t$ as

$$\ln(a_t) + \ln(k_t) = \ln(y_t) - \gamma \ln(g_t)$$

Now, employ the assumption that

$$\ln(k_t) = \psi_1 \ln(k_{t-1}) + \psi_2 \ln(a_t)$$

Using again a tilde to denote log, this assumption allows to write

$$\tilde{a}_t + \tilde{k}_t = (\rho + \psi_2)(\tilde{a}_{t-1} + \tilde{k}_{t-1}) + (\psi_1 - \rho - \psi_2) k_{t-1} + \epsilon_t$$

Due to the assumptions made, $|\psi_1 - \rho - \psi_2| < 1$. Extend:

$$\psi_3 \tilde{k}_{t-1} = \psi_3 \psi_2 (\tilde{a}_{t-2} + \tilde{k}_{t-2}) + \psi_3 (\psi_1 - \psi_2) k_{t-2}$$

and further

$$\psi_3 \tilde{k}_{t-1} = \psi_3 \psi_2 (\tilde{a}_{t-2} + \tilde{k}_{t-2}) + \psi_4 \psi_3 \psi_2 (\tilde{a}_{t-3} + \tilde{k}_{t-3}) + \psi_3 \psi_4^2 \tilde{k}_{t-3}$$

thus:

$$\tilde{a}_t + \tilde{k}_t = (\rho + \psi_2)(\tilde{a}_{t-1} + \tilde{k}_{t-1}) + \sum_{i=0}^{\infty} \psi_2 \psi_3 \psi_4^i (\tilde{a}_{t-2-i} + \tilde{k}_{t-2-i}) + \epsilon_t$$

Due to the assumptions made on $\psi_1$ and $\psi_2$, $\lim_{i \to \infty} \Psi_i = 0$. Thus, the estimated standard error of a large AR of $\ln(a_t) + \ln(k_t)$ can be taken as estimate of $\sigma$. Estimating an AR(8) (suggested by information criteria) yields $\hat{\sigma} = 0.026$.

---

\textsuperscript{15}The investment share of GDP is taken from the IMF’s world economic outlook database.
B Solution of the model under perfect commitment

The FOC for a government with information up to period \( t \) can be written as

\[
E_t[c_{t+1}] \left( \frac{\xi}{\omega} \right) = c_t
\]

Replace \( c_t \) and \( c_{t+1} \) with the budget constraint:

\[
\left( yka_t \right)^{1-\gamma} (y^{-1} - 1) + qB_t - B_{t-1} = \Xi \left( E_t\left[ \left( yka_{t+1} \right)^{1-\gamma} (y^{-1} - 1) + qB_{t+1} \right] - B_t \right)
\]

Rewrite the condition in terms of \( \ln(a_t) \equiv \tilde{a}_t \):

\[
\left( yk \right)^{1-\gamma} (y^{-1} - 1) \exp \left( \frac{1}{1 - \gamma} \tilde{a}_t \right) + qB_t - B_{t-1} = \ldots
\]

\[
\cdots \Xi \left( E_t\left[ \left( yk \right)^{1-\gamma} (y^{-1} - 1) \exp \left( \frac{1}{1 - \gamma} \tilde{a}_{t+1} \right) + qB_{t+1} \right] - B_t \right)
\]

Approximate the equation as

\[
\left( yk \right)^{1-\gamma} (y^{-1} - 1) \left( 1 + \frac{1}{1 - \gamma} \tilde{a}_t \right) + qB_t - B_{t-1} = \ldots
\]

\[
\cdots \Xi \left( \left( yk \right)^{1-\gamma} (y^{-1} - 1) \left( 1 + \frac{1}{1 - \gamma} \tilde{a}_{t+1} \right) + qB_{t+1} \right] - B_t
\]

Write out the expectation for \( \tilde{a}_t \):

\[
\left( yk \right)^{1-\gamma} (y^{-1} - 1) \left( 1 + \frac{1}{1 - \gamma} \tilde{a}_t \right) + qB_t - B_{t-1} = \ldots
\]

\[
\cdots \Xi \left( \left( yk \right)^{1-\gamma} (y^{-1} - 1) \left( 1 + \frac{\rho}{1 - \gamma} \tilde{a}_t \right) + E_t[qB_{t+1}] \right] - B_t
\]

rearrange:

\[
\frac{\left( yk \right)^{1-\gamma} (y^{-1} - 1)(1 - \Xi)}{\omega_1} + \frac{1}{1 - \gamma} \left( \frac{\left( yk \right)^{1-\gamma} (y^{-1} - 1)(1 - \Xi \rho)}{\omega_2} \right) \ldots
\]

\[
\cdots - B_{t-1} = -B_t \left( q + \Xi \right) + \Xi E_t[qB_{t+1}] \quad (8)
\]

Now use the method of undetermined coefficients to find the policy function for \( B_t \). Suppose the policy function in linear in last periods’ debt and the log of technology:

\[
B_t = \alpha_1 + \alpha_2 \tilde{a}_t + \alpha_3 B_{t-1}
\]

If this policy function is correct, the rhs of (8) can be rewritten as

\[
-(q + \Xi) \left( \alpha_1 + \alpha_2 \tilde{a}_t + \alpha_3 B_{t-1} \right) + q\Xi \left( \alpha_1 + \alpha_2 \{ \rho \tilde{a}_t + \epsilon_{t+1} \} + \alpha_3 \{ \alpha_1 + \alpha_2 \tilde{a}_t + \alpha_3 B_{t-1} \} \right)
\]

using \( E_t[\epsilon_{t+1}] = 0 \), expand:

\[
-(q + \Xi) \left( \alpha_1 + \alpha_2 \tilde{a}_t + \alpha_3 B_{t-1} \right) + q\Xi \left( \alpha_1 + \alpha_2 \rho \tilde{a}_t + \alpha_3 \alpha_1 + \alpha_3 \alpha_2 \tilde{a}_t + \alpha_3^2 B_{t-1} \right)
\]

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Collecting the parameters for $B_{t-1}$ one finds

$$-1 = -(q + \Xi)\alpha_3 + q\Xi\alpha_3^2$$

That is

$$\alpha_{3,a,b} = 0.5 \frac{q + \Xi}{\Xi q} \pm 0.5 \sqrt{\frac{(q + \Xi)^2}{\Xi^2 q^2} - 4 \frac{1}{\Xi q}}$$

There are two solutions

$$\alpha_{3,a} = 0.5 \frac{q + \Xi}{\Xi q} + 0.5 \frac{q - \Xi}{\Xi q} = \frac{1}{\Xi} \quad \alpha_{3,b} = 0.5 \frac{q + \Xi}{\Xi q} - 0.5 \frac{q - \Xi}{\Xi q} = \frac{1}{q}$$

Next, collect the parameters for $a_t$ and compute

$$\alpha_2 = \frac{\omega_2}{q\Xi \left( \rho + \alpha_3 - \frac{1}{\Xi} - \frac{1}{q} \right)}$$

Finally, collect the constants and compute

$$\alpha_1 = \frac{\omega_1}{q\Xi \left( 1 + \alpha_3 - \frac{1}{\Xi} - \frac{1}{q} \right)}$$

It is easily verified that both parametrizations for the policy function generate the condition (8).

By recursive substitution, one can rewrite the policy function as function of some past level of debt $B_{t-1}$, the time series of technology and time:

$$B_t = \alpha_1 \frac{1 - \alpha_3^j}{1 - \alpha_3} + \alpha_2 \sum_{j=0}^{l} \alpha_3^j a_{t-j} + \alpha_3^l B_{t-l}$$

Obviously $\alpha_{3,b}$ implies that the level of debt must be explosive independently of the relation between $\beta$ and $q$. This is implausible from an economic point of view since with $\xi = 1$, households are indifferent between borrowing and saving ex-ante. Yet the policy function would imply an explosion of either debt or savings, depending on the parametrization. Thus, this solution is inconsistent with utility maximization, which makes $\alpha_{3,a}$ the unique policy function arising from the solution of households’s maximization problem:

$$B_t = \frac{(yk)^{\frac{1}{1-\gamma}}}{\Xi (q - 1)} \left( \frac{(1 - \Xi)}{(1 - \gamma)(1 - \Xi)} \right) - \tilde{a}_t \frac{1}{\Xi} \left( \frac{1}{1 - \gamma} \right) (\gamma^{-1} - 1) + \frac{1}{\Xi} B_{t-1}$$

C Proofs

C.1 Proof of Proposition 1

For the first statement note that $\frac{\partial \Delta V_1}{\partial B_0} = 0$, while

$$\frac{\partial \Delta U_1}{\partial B_0} = - (y_1 - g_1 - B_0)^{-\eta} + \delta (s(y_1 - g_1 - \delta B_0)^{-\eta}$$

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denote \( B_0 = \varphi(y_1 - g_1) \), which allows to rewrite:
\[
\frac{\partial \Delta U_1}{\partial B_0} = (\kappa(y_1 - g_1))^{-\eta} \left( \delta(\kappa - \delta \varphi)^{-\eta} - (1 - \varphi)^{-\eta} \right)
\]
The derivative is negative if
\[
\delta(\kappa - \delta \varphi)^{-\eta} < (1 - \varphi)^{-\eta} \iff \delta \frac{\eta - 1}{\eta} > \frac{1 - \varphi}{\varphi - \varphi}
\] (9)
If the proposition is true, this must be the case for \( \varphi \in (\frac{1}{1 + \delta}, 1) \). Inspect the borderline cases. In these cases, (9) simplifies to:
\[
\delta \frac{\eta - 1}{\eta} > \begin{cases} \delta & \text{if } \varphi = \frac{1}{1 + \delta} \\ 0 & \text{if } \varphi = 1 \end{cases}
\]
given that \( 0 < \delta < 1 \) and \( \eta > 1 \). Because the rhs of (9) is monotonous in \( \varphi \) in the relevant interval, this also means that \( \frac{\partial \Delta U_1}{\partial B_0} < 0 \), i.e.
an increase of the loan size decreases the utility gain of repayment in period 1 and therewith the probability of a default since the set of innovations that make \( \Delta U_1 < 0 \) increases.

For the second statement start with
\[
\frac{\partial \Delta U_1}{\partial \tilde{a}_0} = (\gamma k)^{-\frac{1}{\gamma - 1}} (\gamma - 1) \frac{\kappa}{\gamma - 1} \eta_1 \left( (y_1 - g_1 - B_0)^{-\eta} - \kappa(y_1 - g_1 - \delta B_0)^{-\eta} \right)
\]
Again, rewrite \( B_0 = \varphi(y_1 - g_1) \). The condition is positive if
\[
(1 - \varphi)^{-\eta} > \kappa(\kappa - \delta \varphi)^{-\eta} \iff \kappa^{-\frac{\eta - 1}{\eta}} > \frac{1 - \varphi}{\varphi - \varphi}
\]
Which holds for the very same reasons as above. Since \( \varphi(\frac{\eta - 1}{\gamma - \gamma}) > 0 \), \( \frac{\partial \Delta U_1}{\partial \tilde{a}_0} > 0 \), implying
\[
\frac{\partial \Delta U_1}{\partial \tilde{a}_0} > 0.
\]

C.2 Proof of Proposition 2

Suppressing the expectations operator for the moment, \( B_0^* \) has the property:
\[
B_0^* = \arg\max\left\{ \frac{1}{1 - \eta} \left( y_0 - g_0 + \beta \xi (1 - d(B_0, \tilde{a}_0))(1 - \delta) B_0 - B_0 \right)^{-\eta} \ldots + \beta \left[ (1 - P(z_1 = 0))u(c_1(z_1 = 1)) + P(z_1 = 0)u(c_1(z_1 = 0)) \right]\right\}
\]
due to perfect competition between lenders. The FOC is
\[
u'(c_0)\left( q_0 - B_0 \beta \xi (1 - \delta) \frac{\partial d(c_1)}{\partial c_0} \right) + \beta \left[ \frac{\partial P(z_1 = 0)}{\partial B_0} u(c_1(z_1 = 0)) - \frac{\partial P(z_1 = 0)}{\partial B_0} u(c_1(z_1 = 1)) \right] \ldots
\]
\[
\ldots + \beta \left[ (1 - P(z_1 = 0))u(c_1(z_1 = 1)) - P(z_1 = 0)u'(c_1(z_1 = 0)) \right] = 0
\]
\[
\Leftrightarrow q_0 - B_0 \beta \xi (1 - \delta) \frac{\partial d(c_1)}{\partial c_0} + \beta \frac{\partial P(z_1 = 0)}{\partial B_0} u(c_1(z_1 = 0)) - u(c_1(z_1 = 1)) \ldots
\]
\[
\ldots = \beta \left[ (1 - P(z_1 = 0))\frac{u'(c_1(z_1 = 1))}{u'(c_0)} + P(z_1 = 0)\frac{u'(c_1(z_1 = 0))}{u'(c_0)} \right]
\] (10)
Now consider the very beginning of the "unsafe" area. While \( P(z_1 = 0) \) is still very close to 0, \( \frac{\partial P(z_1=0)}{\partial B_0} > 0 \). Since \( d_t = P(z_{t+1} = 0) \) information in \( t \) for any point, the partial derivatives must be identical too, allowing to rewrite the approximate FOC in this area as

\[
q_0 - B_0 \delta \xi (1 - \delta) \frac{\partial d(\cdot)}{\partial B_0} + \beta \frac{\partial d(\cdot)}{\partial B_0} \frac{u(c_1(z_1 = 0)) - u(c_1(z_1 = 1))}{u'(c_0)} = \beta \frac{u'(c_1(z_1 = 1))}{u'(c_0)}
\]

If \( \delta \gg 0 \) and \( \kappa \gg 0 \), \( u(c_1(z_1 = 0)) - u(c_1(z_1 = 1)) \approx 0 \) given \( B_0 \) is relatively small. Since \( B_0 > 0 \) (it is not in the "safe" area), the lhs is smaller than \( q_0 \approx \beta \xi \). Not longer suppressing the expectations operator, the approximate FOC then requires that

\[
E_0[u'(c_1)] < \xi u'(c_0) \quad \Leftrightarrow \quad (E_0[y_1 - g_1] - B_0)^{-\eta} - \xi (y_0 - g_0 + q_0 B_0 - B_{-1})^{-\eta}
\]

\[
\Leftrightarrow \quad \xi \frac{\beta}{\eta} (E_0[y_1 - g_1] - B_0) > y_0 - g_0 + q_0 B_0 - B_{-1}
\]

Now consider that lenders design the contract using information from period 0, which allows to restate the requirement as

\[
(yk)^{\frac{1}{\eta}} (y^{-1} - 1) \left( \frac{1}{1 - \gamma} \xi_0 (\Xi \rho - 1) + \Xi - 1 \right) > B_0 (\Xi + q_0) - B_{-1}
\]  \( (11) \)

Since both \( \frac{1}{1 - \gamma} \xi_0 \) and \( \Xi \rho - 1 \) are small numbers, their product is negligible. Furthermore, \( \Xi \approx 1 \), which makes \( \Xi - 1 \approx 0 \). This makes the lhs of (11) approximately zero and gives the simpler requirement

\[
B_{-1} > B_0 (\Xi + q_0) > B_0 (\beta + \xi \delta \beta)
\]

Where \( \xi \delta \beta \) is the smallest possible bond price. If \( d(\cdot) > 0 \), then

\[
B_0 > E_0[y_1 - g_1] \frac{1 - \kappa}{1 - \delta}
\]

Which allows to replace \( B_0 \), yielding the condition.

D On the solution of the full model

D.1 Analytical derivation of \( V_1(\cdot) \)

Begin by rewriting (4) as

\[
V_1(\tilde{\alpha}_t, B_{t-1}) = \max_{\tilde{\alpha}_t} \left\{ u(y_t - g_t + q_t B_t - B_{t-1}) + \beta \left( 1 - d_t \right) E_0[V_1(\tilde{\alpha}_{t+1}, B_t)] \cdots \right.
\]

\[
\cdots + d_t E_0[V_0(\tilde{\alpha}_{t+1}, B_t, 0)] \right\}
\]

To expand further, I abbreviate the notation by denoting the felicity in period \( t \) given access to the capital market as \( u(c_t, $, B_{t-1}) \), the felicity under a fresh default as \( u(c_t, \delta) \) and the felicity for a government that is still excluded from the capital market as \( u(c_t, 0) \). Denote an optimally chosen
$B_t$ as $B_t^*$ and expand $V_1(\tilde{a}_{t+1}, B_t^*)$ and $V_0(\tilde{a}_{t+1}, B_t^*)$. To simplify the notation I shall suppress the expectations operator.

$$V_1(\tilde{a}_t, B_{t-1}) = u(c_t, S, B_{t-1}) + \beta(1 - d_t)u(c_{t+1}, S, B_t^*) + \beta d_t u(c_{t+1}, \delta) \ldots$$

$$\ldots + \beta^2(1 - d_t) \left\{ (1 - d_{t+1})V_1(\tilde{a}_{t+2}, B_{t+1}^*) + d_{t+1}V_0(\tilde{a}_{t+2}, B_{t+1}^*) \right\} \ldots$$

$$\ldots + \beta^2 d_t \left\{ \theta V_1(\tilde{a}_{t+2}, 0) + (1 - \theta)V_0(\tilde{a}_{t+2}, 0) \right\}$$

To avoid a confusion of $V_{0,1}(c_{t+j}, \cdot)$ reached via different time paths, I augment the value functions with the points where the government set $z_t = 0$ for the last time if there are any. I integrate this notational change in the next expansion of value function where I use

$$V_0(\tilde{a}_{t+2}, B_{t+1}^*) = u(c_{t+2}, \delta) + \theta V_1(\tilde{a}_{t+3}, 0, t + 1) + \beta(1 - \theta)V_0(\tilde{a}_{t+3}, 0, t + 1)$$

$$V_0(\tilde{a}_{t+2}, 0) = u(c_{t+2}, 0) + \beta d_t V_1(\tilde{a}_{t+3}, 0, t) + \beta(1 - \theta)V_0(\tilde{a}_{t+3}, 0, t)$$

$$V_1(\tilde{a}_{t+2}, B_{t+1}^*) = u(c_{t+2}, S, B_t^*) + \beta(1 - d_{t+2})V_1(\tilde{a}_{t+3}, B_{t+2}^*) + \beta d_{t+2}V_0(\tilde{a}_{t+3}, B_{t+2}^*)$$

Integrate this into the expression for $V_1(\tilde{a}_t, B_{t-1})$:

$$V_1(\tilde{a}_t, B_{t-1}) = u(c_t, S, B_{t-1}) + \beta(1 - d_t)u(c_{t+1}, S, B_t^*) + \beta d_t u(c_{t+1}, \delta) \ldots$$

$$\ldots + \beta^2(1 - d_t)(1 - d_{t+1})u(c_{t+2}, S, B_{t+1}^*) + \beta^2(1 - d_t)d_{t+1}u(c_{t+2}, \delta) \ldots$$

$$\ldots + \beta^2(1 - \theta)d_t u(c_{t+2}, 0) + \beta^2 \theta d_t u(c_{t+2}, S, 0) \ldots$$

$$\ldots (1 - d_t)(1 - d_{t+1}) \beta^3 \left\{ (1 - d_{t+2})V_1(\tilde{a}_{t+3}, B_{t+2}^*) + d_{t+2}V_0(\tilde{a}_{t+3}, B_{t+2}^*, t + 2) \right\} \ldots$$

$$\ldots + \beta^3(1 - d_t)d_{t+1} \left\{ \theta V_1(\tilde{a}_{t+3}, 0, t + 1) + (1 - \theta)V_0(\tilde{a}_{t+3}, 0, t + 1) \right\} \ldots$$

$$\ldots \beta^3 d_t \theta \left\{ (1 - d_{t+2})V_1(\tilde{a}_{t+3}, B_{t+2}^*, t) + d_{t+2}V_0(\tilde{a}_{t+3}, B_{t+2}^*, t + 2) \right\} \ldots$$

$$\ldots + \beta^3 d_t (1 - \theta) \left\{ \theta V_1(\tilde{a}_{t+3}, 0, t) + (1 - \theta)V_0(\tilde{a}_{t+3}, 0, t) \right\}$$

At a first glance, this expression looks ridiculously bloated and complicated. To understand its structure, concentrate on the multipliers of the felicities in the first three lines. What you see is a record of the first two sets of branches from the tree of possibilities. In the first line, you see the first branch with the possibilities ”repay” and ”default”. In the second line, you see the possibilities that emerge given the government chose to repay in the first branch and in the third line, the possibilities that emerge if the government chose to default. This structure continues into higher branches. Thus, given one has a functional form for $d(\tilde{a}_t, B_t)$ and for the policy function for debt issues, one can easily compute $E_t[u(c_{t+j}), \cdot]$ using numerical integration and thus easily compute both $V_1(\cdot)$ and $V_0(\cdot)$.

Note however a minor difficulty with the potential to completely mess up the results. The standard CRRA felicity function

$$u(c_t) = \frac{c_t^{1-\eta} - 1}{1-\eta}$$

yields negative values if $c_t < 0$. For the calibration I use, $E[c_t] < 1$, which means that about all utility values I compute using a standard CRRA utility function are negative numbers. Now
consider the case $d_t \approx 0$. This means that many paths in the tree of possibilities have zero expected utility. Yet, given negative utility values, these paths dominate all other paths with positive weight in terms of utility. To rule this out, I add a large constant to the CRRA utility which will make sure that all occurring levels of utility are positive.

D.2 On the algorithm

For both $B_{t-1}$ and $\tilde{a}_t$, I construct a grid using 30 Chebyshev nodes. For $\tilde{a}_t$, the grid spans $-0.2$ to $0.2$. For $B_{t-1}$, the grid spans $[-0.2, 3.7]$. As mentioned in the introduction, it is the computation of the values of $V_1(\cdot)$ and $V_0(\cdot)$ that makes my method very slow. To make it feasible at all, I use two tricks:

First, to compute the correct $B_t$, many numerical differentiations of $V_1(\cdot)$ are required. Using $V_1(\cdot)$ is not feasible because a desktop computer is then unable to finish even a single search for $B_t$ within a day. Therefore, I use a 20th order polynomial approximation of $V_1(\cdot)$ in the numerical search for $B_t$, which increases the speed of the computation tremendously without a noteworthy change in the results. This is because, even though $B(\cdot)$ influences $V_1(\cdot)$ and $V_0(\cdot)$, $B_t$ as argmax of $V_1(\cdot)$ is computed given the policy functions and only for this purpose, I use the approximation. The approximation can be estimated via OLS. (This is the reason why I use Chebyshev polynomials / Chebyshev nodes: to prevent singularity issues.)

Second, recall that the weights of different sequences of events consist of $\theta, 1 - \theta, E_t[(1 - d_{t+j})], \text{ and } E_t[d_{t+j}]$. If one of the nondeterministic components is very close to zero, the expected future utilities are numerically irrelevant. I therefore skip the computation of expected utilities once a weight is smaller than 0.001, which improves the speed of a single computation of $V_1(\cdot)$ or $V_2(\cdot)$ by at least 40% but introduces just negligible distortions.